Grade 7/8 Math Circles
March 18-21, 2024
Polynomials - Problem Set

1. Determine if each expression is a polynomial. Explain your reasoning.

   (a) 5 (b) $\frac{2}{x} - x^2$ (c) $x^4 - 6x + 1$ (d) $3^x - 2x$ (e) $x^{2024} + x^{18} + x^3$

   Solution:

   (a) 5 is a polynomial. It has no terms with a variable, therefore it does not violate any rules of polynomials. Hence, any constant number is considered a polynomial.

   (b) Division by our variable in the $\frac{2}{x}$ term indicates that this is not a polynomial.

   (c) This is indeed a polynomial since there is no division by x and there are only whole-number exponents on the variable x (4, 1 and 0).

   (d) Since the variable x is in the exponent, this is not a polynomial.

   (e) All of the exponents are whole numbers and there is no division by the variable x. This is a polynomial.

2. Simplify each polynomial expression.

   (a) $x - 3x^3 + 9x - 4 + 2x^3$ (b) $x^4 - 6x + 1$ (c) $-x - 7x^6 + 5x^4 - 2x^7 - 10x^4 + 5x + x^6$

   Solution:

   (a) We simplify a polynomial expression by combining like-terms. We see $-3x^3$ and $2x^3$ are like-terms since they are both degree 3. Combining them gives $-x^3$. Notice x and $9x$ are also like-terms (1^{st} degree). Combining them gives $10x$. Hence, the simplified expression is $-x^3 + 10x - 4$.

   (b) The expression has no like-terms, so it cannot be simplified further.

   (c) $-7x^6$ and x^6 are like terms. Combining them gives $-6x^6$. Next, $5x^4$ and $-10x^4$ are also like terms. Combining them gives $-5x^4$. Lastly, $-x$ and $5x$ are like-terms. Combining them gives $4x$. The simplified expression is then $-2x^7 - 6x^6 - 5x^4 + 4x$.

3. Recall the expression from part (e) of Question 1: $x^{2024} + x^{18} + x^3$. What is its dominant term

and degree? Is the degree of the overall expression even or odd?

> *Solution*: The dominant term is x^{2024}, which has an even degree. Therefore the entire expression is even-degree.

4. What is the end behaviour of the polynomial function $y = 2x^{17} - 12x^6 + 9$?

> *Solution*: The dominant term is $2x^{17}$, which has a positive leading coefficient (+2) and an *odd* exponent (17). Therefore, this is an odd-degree polynomial. Referring back to the tree diagram from the lesson, an odd-degree polynomial function y with a positive leading coefficient increases as x increases, and decreases as x decreases. Therefore, $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$.

5. Explain whether the following statement is true or false:

   The polynomial function $y = 5x^3 - 2x^2 + x - 15$ may have 0, 1, 2, or 3 real roots, but since it has a degree of 3, it cannot have more than 3 roots.

> *Solution*: The statement is false. It is true that the function cannot have more than 3 roots; however, it cannot have 0 roots since every odd-degree function must cross the x-axis. It can have 1, 2, or 3 real roots.

6. Verify whether the following are roots of the polynomial function $y = 2x^3 + 5x^2 + x - 2$.

   (a) $x = 1$ (b) $x = -1$ (c) $x = \frac{1}{2}$ (d) $x = 2$ (e) $x = -2$

> *Solution*:
>
> (a) Plugging $x = 1$ into the equation $y = 2x^3 + 5x^2 + x - 2$, we get
>
> $$y = 2 \cdot (1)^3 + 5 \cdot (1)^2 + (1) - 2 = 2 + 5 + 1 - 2 = 6$$
>
> This means $y = 6$ when $x = 1$, therefore $x = 1$ *is not* a root of the function.

(b) Plugging $x = -1$ into the equation $y = 2x^3 + 5x^2 + x - 2$, we get

$$y = 2 \cdot (-1)^3 + 5 \cdot (-1)^2 + (-1) - 2 = -2 + 5 - 1 - 2 = 0$$

This means $y = 0$ when $x = -1$, therefore $x = -1$ *is* a root of the function.

(c) Plugging $x = \frac{1}{2}$ into the equation $y = 2x^3 + 5x^2 + x - 2$, we get

$$y = 2 \cdot \left(\frac{1}{2}\right)^3 + 5 \cdot \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right) - 2 = \frac{1}{4} + \frac{5}{4} + \frac{1}{2} - 2 = \frac{6}{4} + \frac{1}{2} - 2 = 2 - 2 = 0$$

This means $y = 0$ when $x = \frac{1}{2}$, therefore $x = \frac{1}{2}$ *is* a root of the function.

(d) Plugging $x = 2$ into the equation $y = 2x^3 + 5x^2 + x - 2$, we get

$$y = 2 \cdot (2)^3 + 5 \cdot (2)^2 + (2) - 2 = 16 + 20 + 2 - 2 = 36$$

This means $y = 36$ when $x = 2$, therefore $x = 2$ *is not* a root of the function.

(e) Plugging $x = -2$ into the equation $y = 2x^3 + 5x^2 + x - 2$, we get

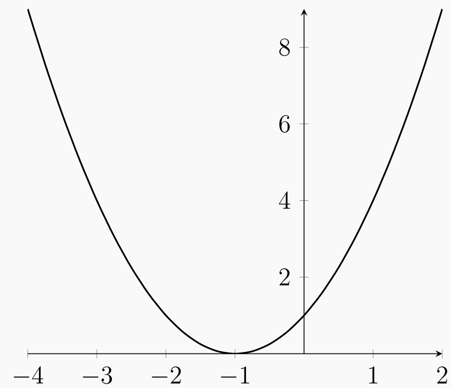
$$y = 2 \cdot (-2)^3 + 5 \cdot (-2)^2 + (-2) - 2 = -16 + 20 - 2 - 2 = 0$$

This means $y = 0$ when $x = -2$, therefore $x = -2$ *is* a root of the function.

7. Consider the function $y = x^2 + 2x + 1$. Solve for the root(s) of the function and explain your result. What might the graph look like?

Solution: The roots of the function $y = x^2 + 2x + 1$ can be found using the quadratic formula. Setting $a = 1$, $b = 2$ and $c = 1$, we plug these values into the formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(1)}}{2(1)} = \frac{-2 \pm \sqrt{4 - 4}}{2} = \frac{-2 \pm \sqrt{0}}{2} = \frac{-2}{2} = -1$$



We see this result in the function plotted above. It just barely skims the x-axis when $x = -1$; in other words, the function has exactly one root at $x = -1$. It then opens upward because of the positive coefficient on the dominant term, x^2.

8. Are the roots of the function $y = -x^2 - 16$ real or imaginary? Explain how you know know without graphing.

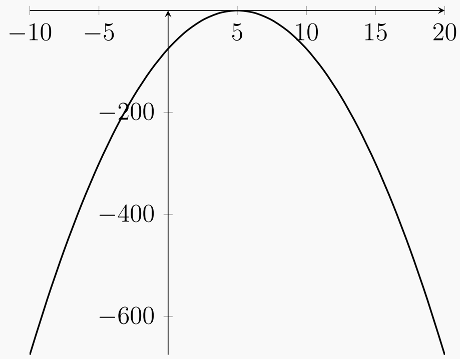
Solution: In order to determine whether a quadratic function's roots are real or imaginary, we must find the discriminant. For the function $y = -x^2 - 16$, we set $a = -1$, $b = 0$ and $c = -1$. The discriminant of a quadratic function is $b^2 - 4ac$. Plugging in our values for a, b and c we see

$$b^2 - 4ac = (0)^2 - 4(-1)(-1) = -4$$

Since the discriminant is negative, the roots of the function are imaginary.

9. The function $y = -3x^2 + 30x - 75$ has a discriminant of zero and a root at $x = 5$. Explain how we can find the maximum height of the graph of the function without graphing.

Solution: If the discriminant of the function is 0, then it has exactly one root. We also notice that the dominant term is $-3x^2$, which has a negative coefficient -3 and an even degree of 2. Using our end behaviour tree diagram from the lesson, we know that y will decrease whether we increase or decrease x. Since the discriminant of the function is zero and the height of the function is 0 at $x = 5$, we know it must have exactly one root at $x = 5$. So, the curve will just barely skim the x-axis at $x = 5$ and will keep decreasing whether x increases or decreases. Its maximum height is then 0. This is shown below.



10. Find the intersection point(s) of the functions $y = 2x^2 - 47x + 9$ and $y = 2x^2 + x + 105$.

Solution: The two functions intersect where their heights are equal. In other words, both function outputs y must equal each other. Therefore,

$$2x^2 - 47x + 9 = 2x^2 + x + 105$$

$$2x^2 - 47x + 9 - 2x^2 - x - 105 = \cancel{2x^2} + \cancel{x} + \cancel{105} - \cancel{2x^2} - \cancel{x} - \cancel{105}$$

$$\implies -48x - 96 = 0$$

$$\frac{\cancel{-48}x}{\cancel{-48}} = \frac{96}{-48}$$

$$x = \frac{98}{-48} = -2$$

This means the two functions intersect when $x = -2$. To find the height at which they intersect, we just plug $x = 2$ into either of the functions. It doesn't matter which function we choose because if they intersect at $x = -2$, their heights should be the same! Let's choose $y = 2x^2 - 47x + 9$. We have $y = 2(-2)^2 - 47(-2) + 9 = 8 + 94 + 9 = 111$, so the two graphs intersect at the coordinate $(x, y) = (-2, 111)$.

11. Two sports equipment companies compete to see who can generate the most revenue. Company 1 generates a total revenue R_1 (in thousands of dollars) after each month t based on the function

$$R_1 = 7t^3 + 5t^2 - t - 2$$

while Company 2 generates revenue in thousands of dollars based on the function

$$R_2 = 7t^3 - t^2 + 4t - 1$$

At what time do the two companies generate the exact same revenue?

Solution: The two companies generate the exact same revenue when $R_1 = R_2$. Setting these two equations equal to each other, we get

$$7t^3 + 5t^2 - t - 2 = 7t^3 - t^2 + 4t - 1$$

We see that we can cancel the $7t^3$ terms from both sides, leaving us with the equation

$$5t^2 - t - 2 = -t^2 + 4t - 1$$

We now want to solve for t, so we need to use the quadratic formula. In order to use the formula, one side of the equation must equal zero. Let's move everything from the right side over to the left.

$$5t^2 - t - 2 + t^2 - 4t + 1 = \cancel{-t^2} + \cancel{4t} - \cancel{1} + \cancel{t^2} - \cancel{4t} + \cancel{1}$$

$$6t^2 - 5t - 1 = 0$$

We can now use the quadratic formula by setting $a = 6$, $b = -5$ and $c = -1$. We have

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(6)(-1)}}{2(6)} = \frac{5 \pm \sqrt{25 + 24}}{12} = \frac{5 \pm \sqrt{49}}{12} = \frac{5 \pm 7}{12}$$

We are only considering positive values of time, so we ignore the case where we subtract. Hence, the time in which the two companies' revenues will be equivalent is when $t = \frac{5+7}{12} = \frac{12}{12} = 1$ month.

12. * The Pascal Trading Company sells hockey cards at sports conventions. They price all of their cards at \$5 each. At this price, they can expect to sell 165 cards each day. However, every time they increase the price of their cards by \$1, they sell an average of 15 less cards per day.

    (a) How should the company price each card to generate the maximum possible revenue?

(*Note:* The total revenue is found by multiplying the number of products by the cost of each product.)

(b) Each convention is one week long. What is the maximum revenue for the Pascal Trading Company during each convention?

Solution:

(a) Using the note given, the revenue R can be modeled by the formula

$$R = (\text{cost of each product}) \cdot (\#\text{ of products})$$

With no price changes, the company's daily revenue is just

$$R = (\$5) \cdot (165) = \$825$$

We're told that every time the price of the \$5 cards is raised by \$1, there are 15 fewer cards sold. If we let x represent the number of times the price is raised by \$1, the equation for the company's daily revenue is

$$R = (5 + x) \cdot (165 - 15x)$$

If we graph this function, we can see that it has a **maximum** value. The x-value that corresponds to this maximum is exactly between the first and second root of the function. To find the roots of the function, we set the revenue $R = 0$. Therefore

$$0 = (5 + x) \cdot (165 - 15x)$$

This means that either (i) $5 + x = 0$ or (ii) $165 - 15x = 0$. For (i), we see that

$$5 + x = 0 \implies x = -5$$

so the first root of the function is $x = -5$. From (ii), the second root of the function is found by solving

$$165 - 15x = 0$$
$$\frac{165}{15} = \frac{\cancel{15}x}{\cancel{15}}$$

$$\implies x = \frac{165}{15} = 11$$

so the second root of the function is $x = 11$.

We can next imagine a mid-line being drawn down the middle of the curve, splitting it exactly in half. Since this is a *quadratic function*, the curve is symmetric, which means the maximum value will occur when the x-value is in the middle of these two roots. The two roots are $x = -5$ and $x = 11$, so the maximum value of the function will occur at the average of these two values (when $x = \frac{-5+11}{2} = \frac{6}{2} = 3$).

Now that we know the maximum revenue occurs when $x = 3$, let's remember that x represented the number of times the company increased the price of each card by \$1. If $x = 3$, then the company increased the price of each card by 1 a total of three times, giving us a total price increase of \$3 and a new price of \$5 + \$3 = \$8.

(b) The maximum revenue R_{max} is achieved when $x = 3$. We simply plug $x = 3$ into our revenue formula

$$R = (5 + x) \cdot (165 - 15x)$$

$$\implies R_{\text{max}} = (5 + 3) \cdot (165 - 15 \cdot 3)$$

$$= (8) \cdot (165 - 45)$$

$$= (8) \cdot (120)$$

$$\therefore R_{\text{max}} = 960$$

Hence, the maximum daily revenue the company earns is \$960, and occurs when the price of each card is increased to \$8. If the convention is one week (7 days) long, we multiply the daily revenue by 7 to find the the maximum weekly revenue is $\$960 \cdot 7 = \$6,720$.